

Figure 1. Schematic of a finite plate subject to turbulent and mean flow excitations.

- 22 Routh, E. J., *A Treatise on the Stability of a Given State of Motion*, Macmillan & Co., London, 1877.
- 23 Parks, P. C., “A new proof of the Routh-Hurwitz stability criterion using the second method of Lyapunov,” *Proceedings of the Cambridge Philosophical Society*, Vol. 58, 1962, pp. 694 – 702.
- 24 Hardin, J. C., *Introduction to Time Series Analysis*, NASA Reference Publication 1145, 1990.

11. Gedney, C. J. and Leehey, P., "Measurements of the low wavenumber wall pressure spectral density during transition on a flat plate," Acoustics and Vibration Laboratory, MIT, Report 93019-1, 1984.
12. Farabee, T. M. and Casarella, M. J., "Spectral features of wall pressure fluctuations beneath turbulent boundary layers," *Physics of Fluids*, Vol. 3, 1991, pp. 2410 – 2420.
13. Maestrello, L., Frendi, A., and Brown, D. E., "Nonlinear vibration and radiation from a panel with transition to chaos induced by acoustic waves," *AIAA Journal*, Vol. 30, 1992, pp. 2632 – 2638.
14. Frendi, A., Maestrello, L., and Bayliss, A., "Coupling between a supersonic boundary layer and a flexible surface," *AIAA Journal*, Vol. 31, 1993, pp. 708 – 713.
15. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, Inc., 1965.
16. Curle, N., "The influence of solid boundaries upon aerodynamic sound," *Proceedings of the Royal Society of London*, Vol. 231, 1955, pp. 505 – 514.
17. Powell, A., "Aerodynamic noise and the plane boundary," *Journal of the Acoustical Society of America*, Vol. 32, 1960, pp. 982 – 990.
18. Lighthill, M. J., "On sound generated aerodynamically. I. General theory," *Proceedings of the Royal Society of London*, Vol. 211, 1952, pp. 564 – 587.
19. Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.
20. Dowell, E. H., "Flutter of a buckled plate as an example of chaotic motion of a deterministic autonomous system," *Journal of Sound and Vibration*, Vol. 85, 1982, pp. 333 – 344.
21. Schwarz, R. J., and Friedland, B., *Linear Systems*, McGraw-Hill Book Co., New York, 1965.

References

1. Ludwig, G. R., "An experimental investigation of the sound generated by thin steel panels excited by turbulent flow (boundary layer noise)," U.T.I.A. Report 87, 1987.
2. Corcos, G. M., "Resolution of pressure in turbulence," *Journal of the Acoustical Society of America*, Vol. 35, 1963, pp. 192 – 199.
3. Corcos, G. M., "The resolution of turbulent pressure at the wall of a boundary layer," *Journal Sound and Vibration*, Vol. 6, 1967, pp. 59 – 70.
4. Maestrello, L., "Design criterion of panel structure excited by turbulent boundary layer," *Journal of Aircraft*, Vol. 5, 1968, pp. 321 – 328.
5. Maestrello, L., "Radiation from and panel response to a supersonic turbulent boundary layer," *Journal of Sound Vibration* Vol. 10, 1969, pp. 261 – 295.
6. Maestrello, L. and Linden, T. L. J., "Response of an acoustically loaded panel excited by supersonically convected turbulence," *Journal of Sound and Vibration*, Vol. 16, 1971, pp. 365 – 384.
7. Farabee, T. M. and Geib, E. F., "Measurement of boundary layer pressure fields with an array of pressure transducers in a subsonic flow," DTNSRDC Rep. No. 76-0031, 1976.
8. Martin, N. C. and Leehey, P., "Low wavenumber wall pressure measurements using a rectangular membrane as a spatial filter," *Journal of Sound and Vibration*, Vol. 52, 1977, pp. 95 – 120.
9. Yen, D. H., Maestrello, L., and Padula, S. L., "Response of a panel to a supersonic turbulent boundary layer: Studies on a theoretical model," *Journal of Sound and Vibration*, Vol. 71, 1980, pp. 271 – 282.
10. Ffowcs Williams, J. E., "Boundary-layer pressures and the Corcos model: a development to incorporate low-wavenumber constraints," *Journal of Fluid Mechanics*, Vol. 125, 1982, pp. 9 – 25.

be found. The formulation given by Eq. (63) represents a new way of estimating the radiated acoustic pressure from an elastic plate excited by turbulent flow. In deriving Eq. (63), the effects of coupling among structural modes, radiated acoustic pressure, and turbulent flow are all taken into account.

VI. CONCLUDING REMARKS

The cross-power spectral density function of the radiated acoustic pressure from a finite plate subject to turbulent flow is obtained and expressed in terms of the cross-power spectral density function of the turbulent boundary layer in an explicit form. In deriving this formulation, the effects of stretching due to in-plane force and those of coupling between structural vibration and acoustic radiation are taken into account. A general stability analysis is given by using the basic existence-uniqueness theorem. In particular, the stable conditions for a linearized system are derived by using the Routh algorithm. Two sources of instabilities are found to be attributable to the added damping and stiffness due to fluid loading and flow. The effects of the added damping and stiffness increase with the Mach number of the mean flow, the former increasing linearly and the latter increasing quadratically.

ACKNOWLEDGEMENTS

This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-19480 while the first author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23681.

where $\xi' = x_1 - x_2$, $\eta' = y_1 - y_2$, and $\tau' = t_1 - t_2$. In carrying out the integration with respect to t_2 in Eq. (61), we have used the definition of the cross-correlation function Γ_w (see Eq. (55)) and the property of the Dirac delta function.²⁴

Substituting Eq. (61) into (60) and using the definition of the cross-power spectrum for \mathcal{P}_w , we obtain

$$\begin{aligned} \langle p(x, y, z, t) p^*(x', y', z', t') \rangle &= \frac{\rho^2}{(2\pi)^4} \int_0^b \int_0^b \int_0^L \int_0^L \int_{-\infty}^{\infty} \\ &\quad \left(-i\omega + U \frac{\partial}{\partial x} \right) \left(i\omega' + U \frac{\partial}{\partial x'} \right) G(x, y, z | x_1, y_1, 0) G^*(x', y', z' | x_2, y_2, 0) \\ &\quad \times \left(-i\omega + U \frac{\partial}{\partial x_1} \right) \left(i\omega + U \frac{\partial}{\partial x_2} \right) \mathcal{P}_w(\xi', \eta', \omega) e^{i\omega\tau} d\omega dx_1 dx_2 dy_1 dy_2, \end{aligned} \quad (62)$$

where $\mathcal{P}_w(\xi', \eta', \omega)$ is given by Eq. (56).

The cross-power spectral density function for the radiated acoustic pressure can now be obtained by taking a Fourier transformation of Eq. (62). Using the property of the Dirac delta function, we can carry out integrations with respect to τ and ω exactly and obtain

$$\begin{aligned} \mathcal{P} &= \frac{\rho^2}{(2\pi)^4} \int_0^b \int_0^b \int_0^L \int_0^L \\ &\quad \left(-i\omega + U \frac{\partial}{\partial x} \right) \left(i\omega' + U \frac{\partial}{\partial x'} \right) G(x, y, z | x_1, y_1, 0) G^*(x', y', z' | x_2, y_2, 0) \\ &\quad \times \left(-i\omega + U \frac{\partial}{\partial x_1} \right) \left(i\omega + U \frac{\partial}{\partial x_2} \right) \mathcal{P}_w(\xi', \eta', \omega) dx_1 dx_2 dy_1 dy_2. \end{aligned} \quad (63)$$

Equation (63) shows that the cross-power spectral density function of the radiated acoustic pressure is directly related to the cross-power spectral density function of the plate flexural displacement. If we substitute Eq. (56) into (63), we can obtain an explicit formulation for \mathcal{P} in terms of the cross-power spectral density function of turbulent flow.

It is emphasized here that turbulent flow is random in nature, and therefore deterministic formulations for calculating the resulting structural and acoustic responses cannot

The frequency domain solution for the radiated acoustic pressure \hat{p} can be obtained by substituting Eq. (7) into (13)

$$\begin{aligned} \hat{p}(x, y, z, \omega) = \frac{i\rho}{(2\pi)^2} \left(-i\omega + U \frac{\partial}{\partial x} \right) \int_0^b \int_0^L G(x, y, z | x_1, y_1, 0) \\ \times \left(-i\omega + U \frac{\partial}{\partial x_1} \right) \hat{w}(x_1, y_1, \omega) dx_1 dy_1. \end{aligned} \quad (59a)$$

In a similar manner, we can write

$$\begin{aligned} \hat{p}^*(x', y', z', \omega') = -\frac{i\rho}{(2\pi)^2} \left(i\omega' + U \frac{\partial}{\partial x'} \right) \int_0^b \int_0^L G^*(x', y', z' | x_2, y_2, 0) \\ \times \left(i\omega' + U \frac{\partial}{\partial x_2} \right) \hat{w}^*(x_2, y_2, \omega') dx_2 dy_2, \end{aligned} \quad (59b)$$

where G^* is the complex conjugate of G defined in Eq.(9) for a supersonic flow or (11) for a subsonic flow.

Substituting Eq. (59) into (58) and rearranging terms, we obtain

$$\begin{aligned} \langle p(x, y, z, t) p^*(x', y', z', t') \rangle = \frac{\rho^2}{(2\pi)^4} \int_0^b \int_0^b \int_0^L \int_0^L \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \\ \left(-i\omega + U \frac{\partial}{\partial x} \right) \left(i\omega' + U \frac{\partial}{\partial x'} \right) G(x, y, z | x_1, y_1, 0) G^*(x', y', z' | x_2, y_2, 0) \\ \times \left(-i\omega + U \frac{\partial}{\partial x_1} \right) \left(i\omega' + U \frac{\partial}{\partial x_2} \right) \langle \hat{w}(x_1, y_1, \omega) \hat{w}^*(x_2, y_2, \omega') \rangle \\ \times e^{-i\omega t} e^{i\omega' t'} d\omega d\omega' dx_1 dx_2 dy_1 dy_2, \end{aligned} \quad (60)$$

where the triangle-bracketed term on the right side of Eq. (60) can be rewritten as

$$\begin{aligned} \langle \hat{w}(x_1, y_1, \omega) \hat{w}^*(x_2, y_2, \omega') \rangle \\ = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle w(x_1, y_1, t_1) w^*(x_2, y_2, t_2) \rangle e^{-i\omega t_1} e^{i\omega' t_2} dt_1 dt_2 \\ = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_w(\xi', \eta', \tau') e^{-i\omega' \tau'} e^{-i(\omega - \omega') t_2} dt_2 d\tau' \\ = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \Gamma_w(\xi', \eta', \tau') e^{-i\omega' \tau'} \delta(\omega - \omega') d\tau', \end{aligned} \quad (61)$$

Second, we expand $w(x, y, t)$ into the orthogonal base functions using Eq. (26) and replace the coefficient matrix $[C(t)]$ by the solution matrix $[\mathcal{E}]$. Doing so leads to

$$\Gamma_w(\xi, \eta, \tau) = \{W(x)\}^T [\mathcal{E}] [F] [\mathcal{E}^*]^T \{W(x')\} \quad (55)$$

The corresponding cross-power spectrum of the plate flexural displacement is thus found to be

$$\begin{aligned} \mathcal{P}_w(\xi, \eta, \omega) &= \{W(x)\}^T [\mathcal{E}] \\ &\times \left[\sum_k \sum_l \int_0^L \int_0^L \{W(x_0)\}^T \mathcal{P}_{f,kl}(\xi_0, \eta, 0, \omega) \{W(x'_0)\} dx'_0 dx_0 \right] [\mathcal{E}^*]^T \{W(x')\}, \end{aligned} \quad (56)$$

where $\mathcal{P}_{f,kl}(\xi, \eta, 0, \omega)$ is given by Eq. (53).

Equation (56) demonstrates that the power spectral density function of the plate response is directly related to the power spectral density function of the excitation forcing field due to a turbulent boundary layer. The auto-power spectral density of the plate flexural displacement can be obtained simply by setting $\xi = \eta = 0$ in Eq. (56).

V. RADIATED ACOUSTIC PRESSURE

In this section, we derive a formulation for estimating the power spectral density function of the radiated acoustic pressure based on plate flexural vibration responses. First, let us define a cross-correlation function for the radiated acoustic pressures measured at two different points in the fluid medium

$$\Gamma = \langle p(x, y, z, t) p^*(x', y', z', t') \rangle, \quad (57)$$

where p^* is the complex conjugate of p given by Eq. (14) or (17), depending upon whether the flow is supersonic or subsonic. Next, we replace p in Eq. (57) by its Fourier transformation pair

$$\langle p(x, y, z, t) p^*(x', y', z', t') \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \hat{p}(x, y, z, \omega) \hat{p}^*(x', y', z', \omega') \rangle e^{i\omega t} e^{-i\omega' t'} d\omega d\omega'. \quad (58)$$

The corresponding cross-power spectrum can be obtained by taking a Fourier transformation of $\Gamma_f(\xi, \eta, 0, \tau)$. Evaluations of this Fourier transformation can be facilitated by the residue theory, and the result is

$$\mathcal{P}_f(\xi, \eta, 0, \omega) = \langle \overline{p^2} \rangle (\delta/U_e) e^{-|\xi|/(a_1\delta)} e^{-|\eta|/(a_2\delta)} e^{-i\xi\omega/U} \sum_{j=1}^4 A_j e^{-iK_j\omega\delta/U}. \quad (49)$$

Next, we rewrite the ensemble average of the cross correlation of the forcing field Γ_f with p_T expanded in terms of the base functions

$$\Gamma_f(\xi, \eta, 0, \tau) = \{W(x)\}^T [F] \{W(x')\}, \quad (50)$$

where $[F]$ is the cross-correlation matrix defined as

$$[F] = \langle [f(t)] \{\tilde{W}(y)\} \{\tilde{W}(y')\}^T [f^*(t')]^T \rangle. \quad (51)$$

The corresponding cross-power spectral density function can be obtained by taking a Fourier transformation of Eq. (50)

$$\mathcal{P}_f(\xi, \eta, 0, \omega) = \{W(x)\}^T \left\langle \frac{1}{2\pi} \int_{-\infty}^{\infty} [F] e^{-i\omega\tau} d\tau \right\rangle \{W(x')\}. \quad (52)$$

Equating Eq. (52) to (49) and using the orthogonality properties of the base functions, we can obtain the cross-power spectral density matrix whose (m, n) th element is given by

$$\mathcal{P}_{f,kl} = \int_0^L \int_0^L \langle \overline{p^2} \rangle \sum_{q=1}^4 \left\{ \frac{2A_q K_q e^{-|\xi|/(a_1\delta)} e^{-|\eta|/(a_2\delta)}}{K_q^2 + [U_e/(U\delta)]^2 (\xi - U\tau)^2} \right\} W_k(x) W_l(x') dx' dx. \quad (53)$$

Once this is done, we can proceed to derive a cross-power spectral density function for the plate flexural vibration response. First, let us define a cross-correlation function for the plate flexural displacement

$$\Gamma_w(\xi, \eta, \tau) = \langle w(x, y, t) w^*(x', y', t') \rangle. \quad (54)$$

It has been shown that the plate flexural displacement $w(x, y, t)$ can be expanded in terms of the orthogonal base functions $\{W(x)\}^T [C(t)] \{\tilde{W}(y)\}$ along the x - and y -axes, respectively, where the coefficients $[C(t)]$ are determined by Eq. (28). For convenience in the derivation, let us assume that $[C(t)]$ can be expressed in terms of a solution matrix $[\mathcal{E}]$

$$[C(t)] = [\mathcal{E}][f(t)] \quad (47)$$

where $[\mathcal{E}]$ is independent of time t and can be determined numerically, $[f(t)]$ represents the forcing excitation due to a turbulent flow. The elements of each of the sub-matrices $[f^{ij}(t)]$ are given by Eq. (30).

Since the turbulent boundary layer is random in nature, one must rely on the statistical properties of the turbulent flow. However, in many applications a turbulent boundary layer can be assumed to be stationary in time and homogeneous in space, so that it can be described by a space-time cross-correlation function which decays with spatial and temporal separations and convects with flow at velocity U . Based on the mathematical model developed by Maestrello,^{4, 5} we can express the ensemble average of the cross correlation of the excitation forcing field due to a turbulent boundary layer as

$$\begin{aligned} \langle p_T(x, y, 0, t) p_T^*(x', y', 0, t') \rangle &\equiv \Gamma_f(\xi, \eta, 0, \tau) \\ &= \langle \overline{p^2} \rangle \sum_{q=1}^4 \left\{ \frac{2A_q K_q e^{-|\xi|/(a_1 \delta)} e^{-|\eta|/(a_2 \delta)}}{K_q^2 + [U_e/(U\delta)]^2 (\xi - U\tau)^2} \right\} \end{aligned} \quad (48)$$

where superscript $*$ indicates a complex conjugation, $\xi = x - x'$, $\eta = y - y'$, $\tau = t - t'$, δ is the boundary layer thickness, $A_1 = 0.044$, $A_2 = 0.075$, $A_3 = -0.093$, $A_4 = -0.025$, $K_1 = 0.0578$, $K_2 = 0.243$, $K_3 = 1.12$, $K_4 = 11.57$, $a_1 = 50/(\overline{C_f R_\theta})$, here $\overline{C_f R_\theta}$ is the equivalent incompressible Reynold's number, $a_2 = 0.26$, U_e is the free stream velocity, and $\langle \overline{p^2} \rangle$ is a measure of the mean square intensity of the forcing field.

Consequently, the linearized system, Eq. (41), is stable when all four conditions given by Eq. (45) are met. As an example, let us consider the first two conditions. Substituting Eq. (44) into (45) yields

$$\frac{\Phi_{11}\Psi_{22} + \Phi_{22}\Psi_{11} - \Phi_{12}\Psi_{21} - \Phi_{21}\Psi_{12}}{\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}} > 0 \quad (46a)$$

$$\frac{\chi_{11}\chi_{22} - \chi_{12}\chi_{21}}{\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}} > 0. \quad (46b)$$

where Φ_{ij} , Ψ_{ij} , and χ_{ij} represent the mass, damping, and stiffness per unit area of the plate and are defined in Eq. (29). In particular, Φ_{ij} includes the added mass, Ψ_{ij} includes the added damping, and χ_{ij} includes the added stiffness induced by acoustic radiation. If the effects of fluid loading and mean flow are neglected, then $\Phi_{ij} = \Psi_{ij} \equiv 0$, for $i \neq j$, so the inequality in Eq. (46a) is automatically satisfied. Also, since the stiffness matrix is predominately a diagonal matrix, the inequality in Eq. (46b) always holds true. Thus without fluid loading and mean flow, a linear system is always stable. Equation (46) thus shows that when the effects of fluid loading and mean flow are large enough to change one of the inequalities given by Eq. (46), then the amplitude of the plate flexural vibration may grow exponentially without a bound.

One advantage of the stability analysis discussed above is that it allows one to relate the structural instability phenomenon directly to the physical quantities possessed by an elastic structure. In a following paper, we will use this stability analysis to analyze the mechanisms of structural instabilities under various fluid loading and mean flow conditions. In particular, we will plot stable charts so that we know when and why an elastic structure may become unstable.

IV. EXCITATION FORCING FIELD

In the preceding section, we presented a general stability analysis and, in particular, derived stable conditions for a linearized system. In this section, we consider the steady-state response of a finite plate excited by a turbulent boundary layer.

Without loss of generality, we can assume a form of solution for $\{Y\}$ as

$$\{Y\} = \{\overline{Y}\}e^{\Lambda t} \quad (42)$$

where $\{\overline{Y}\}$ represents the amplitude of $\{Y\}$.

Substituting Eq. (42) into (41), we can derive the characteristic equation for the eigenvalue Λ

$$\sum_{j=0}^4 \Omega_j \Lambda^j = 0 \quad (43)$$

where Ω_j are given by

$$\Omega_0 = \beta_{11}\beta_{22} - \beta_{12}\beta_{21} \quad (44a)$$

$$\Omega_1 = \alpha_{11}\beta_{22} + \alpha_{22}\beta_{11} - \alpha_{12}\beta_{21} - \alpha_{21}\beta_{12} \quad (44b)$$

$$\Omega_2 = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} - \beta_{11} - \beta_{22} \quad (44c)$$

$$\Omega_3 = -\alpha_{11} - \alpha_{22} \quad (44d)$$

$$\Omega_4 = 1. \quad (44e)$$

The stability theorem for linear systems²¹ states that a linear system is stable if and only if the roots of the characteristic equation all lie in the left half-plane, excluding the imaginary axis. To determine whether the polynomial given by Eq. (43) has all its roots in the left half-plane without actually solving for all the roots, we use the Routh table,^{22, 23} and derive four parameters for the polynomial given by Eq. (43). For the roots of this polynomial to be confined in the left half-plane, excluding the imaginary axis, these four parameters must all be strictly positive. Such a requirement yields the following four conditions

$$\Omega_0 > 0 \quad (45a)$$

$$\Omega_3 > 0 \quad (45b)$$

$$\Omega_2\Omega_3 - \Omega_1 > 0 \quad (45c)$$

$$\Omega_1(\Omega_2\Omega_3 - \Omega_1) - \Omega_0\Omega_3^2 > 0. \quad (45d)$$

Substituting Eq. (36) into (34), we obtain

$$\{\dot{Y}\} = (\mathcal{J}[\mathcal{A}] + \mathcal{J}[\mathcal{B}]) \{Y\} \quad (37)$$

where $\{\dot{Y}\}$ is the time derivative of $\{Y\}$ given by $\{\dot{Y}\}^T = \{\dot{Y}_1, \dot{Y}_2, \dot{Y}_3, \dot{Y}_4\}$, $\mathcal{J}[\mathcal{A}]$ is the Jacobian of the linear part of Eq. (34) defined by

$$\mathcal{J}[\mathcal{A}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \alpha_{12} \\ 0 & 0 & 0 & 1 \\ \beta_{21} & \alpha_{21} & \beta_{22} & \alpha_{22} \end{bmatrix} \quad (38)$$

where α_{kl} and β_{kl} are given by Eq. (35). Similarly, $\mathcal{J}[\mathcal{B}]$ in Eq. (37) is the Jacobian of the nonlinear part of Eq. (34) defined by

$$\mathcal{J}[\mathcal{B}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (\epsilon_{11}Y_1 + \zeta_{11}Y_3)^2 & 0 & (\epsilon_{12}Y_1 + \zeta_{12}Y_3)^2 & 0 \\ 0 & 0 & 0 & 0 \\ (\epsilon_{21}Y_1 + \zeta_{21}Y_3)^2 & 0 & (\epsilon_{22}Y_1 + \zeta_{22}Y_3)^2 & 0 \end{bmatrix} \quad (39)$$

where ϵ_{kl} and ζ_{kl} are given by Eq. (33).

The stabilities of the system defined by Eq. (37) can now be examined by solving for the equilibria obtained by setting the left side of Eq. (37) to zero¹⁹

$$(\mathcal{J}[\mathcal{A}] + \mathcal{J}[\mathcal{B}]) \{Y\} = 0. \quad (40)$$

In general, for a nonlinear system with cubic nonlinearities there are multiple equilibrium positions,²⁰ and the plate flexural vibration may become chaotic self-excited oscillations in the presence of mean flow.

For simplicity, we demonstrate in what follows a stability analysis of a linearized system obtained by omitting the nonlinearities in Eq. (37)

$$\{\dot{Y}\} = \mathcal{J}[\mathcal{A}]\{Y\}. \quad (41)$$

where the elements of matrices $[\Phi]$, $[\Psi]$, $[\chi]$, and $[\Xi]$ are given by Eqs. (29), with the superscripts omitted for brevity. In particular, the elements of the matrix $[\Xi]$ are quadratic in C and can be written as

$$\Xi_{kl} = (\epsilon_{kl}C_1 + \zeta_{kl}C_2)^2 \quad (32)$$

where the elements of ϵ_{kl} and ζ_{kl} are defined as

$$\epsilon_{kl} = \sqrt{\frac{Eh}{2bL} \int_0^L W_k''(x)W_l(x)dx \int_0^L W_1'^2(x')dx' \int_0^b \tilde{W}_1^4(y)dy} \quad (33a)$$

$$\zeta_{kl} = \sqrt{\frac{Eh}{2bL} \int_0^L W_k''(x)W_l(x)dx \int_0^L W_2'^2(x)dx \int_0^b \tilde{W}_1^4(y)dy} \quad (33b)$$

with indices k and l varying from 1 to 2.

Now we use the basic existence-uniqueness theorem¹⁹ to analyze the stability of the system defined by Eq. (31). First, let us rewrite Eq. (31) in the following way

$$\{\ddot{C}\} = [\alpha]\{\dot{C}\} + [\beta]\{C\} + [\nu]\{C\} \quad (34)$$

where matrices $[\alpha]$, $[\beta]$, and $[\nu]$ are defined as

$$[\alpha] = -[\Phi]^{-1}[\Psi] \quad (35a)$$

$$[\beta] = -[\Phi]^{-1}[\chi] \quad (35b)$$

$$[\nu] = [\Phi]^{-1}[\Xi] \quad (35c)$$

where $[\Phi]^{-1}$ represents the inversion of the mass matrix.

Next, we define new variables

$$Y_1 = C_1 \quad (36a)$$

$$Y_2 = \dot{C}_1 \quad (36b)$$

$$Y_3 = C_2 \quad (36c)$$

$$Y_4 = \dot{C}_2. \quad (36d)$$

powers of the unknown coefficients $\{C^j\}$. Note that in this case coupling occurs not only through nonlinearities, but also through added mass, added damping, and added stiffness due to acoustic radiation.

The elements of the sub-matrices $[f^{ij}]$ on the right side of Eq. (21) are given by

$$f_{kl}^{ij} = -\delta_{kl}f(t). \quad (30)$$

Note that in order to simplify the problem, we have assumed that the time delay Δt is negligibly small in the derivation of Eq. (28). This assumption is permissible if the plate has a finite size and if the time required for an acoustic signal to traverse the plate surface is substantially less than the typical period of the signal.

III. STABILITY ANALYSIS

Equation (28) represents a system of coupled, nonlinear, integral equations. Because of the presence of nonlinearities and coupling between structural vibration and acoustic radiation, the amplitude of the plate flexural vibration may become unstable under certain circumstances. Therefore a general stability analysis is given below.

Equation (28) shows that for N longitudinal modes and M transverse modes, one need solve an $(N \times M) \times (N \times M)$ matrix equation. Solutions to such a matrix equation are extremely involved. For the purpose of illustration, let us consider only one mode in the y -axis direction and two modes in the x -axis direction. Thus, Eq. (28) reduces to a 2×2 matrix equation. The stability of such a system can be analyzed by examining the homogeneous part of that equation

$$[\Phi]\{\ddot{C}\} + [\Psi]\{\dot{C}\} + [\chi]\{C\} - [\Xi]\{C\} = 0 \quad (31)$$

where $[\Phi]$, $[\Psi]$, and $[\chi]$ consist of sub-matrices $[\Phi^{ij}]$, $[\Psi^{ij}]$, and $[\chi^{ij}]$, respectively, which represent the effects of coupling between all the modes in the x -axis direction and the j th mode in the y -axis direction. The elements of each of the sub-matrices $[\Phi^{ij}]$, $[\Psi^{ij}]$, $[\chi^{ij}]$, and $[\Xi^{ij}]$ can be written in the following general forms

$$\begin{aligned} \Phi_{kl}^{ij} = & \rho_p h \delta_{ij} \delta_{kl} \\ & + i \frac{\rho_0}{(2\pi)^2 b L} \int_0^b \int_0^L \int_0^b \int_0^L \frac{(1 + U\Theta)}{R} \tilde{W}_i(y') \tilde{W}_j(y) W_k(x') W_l(x) dx' dy' dx dy \end{aligned} \quad (29a)$$

$$\begin{aligned} \Psi_{kl}^{ij} = & \gamma \delta_{ij} \delta_{kl} - i \frac{\rho_0 U}{(2\pi)^2 b L} \int_0^b \int_0^L \int_0^b \int_0^L \\ & \left[\frac{\Upsilon}{R} W_k(x') - \frac{(1 + U\Theta)}{R} \frac{\partial W_k(x')}{\partial x'} \right] \tilde{W}_i(y') \tilde{W}_j(y) W_l(x) dx' dy' dx dy \end{aligned} \quad (29b)$$

$$\begin{aligned} \chi_{kl}^{ij} = & D \left\{ \left[\left(\frac{\lambda_i}{b} \right)^4 + \left(\frac{\lambda_k}{L} \right)^4 \right] \delta_{ij} \delta_{kl} + \frac{2}{bL} \int_0^b \tilde{W}_i''(y) \tilde{W}_j(y) dy \int_0^L W_k''(x) W_l(x) dx \right\} \\ & + i \frac{\rho_0 U^2}{(2\pi)^2 b L} \int_0^b \int_0^L \int_0^b \int_0^L \frac{\Upsilon}{R} \frac{\partial W_k(x')}{\partial x'} \tilde{W}_i(y') \tilde{W}_j(y) W_l(x) dx' dy' dx dy \end{aligned} \quad (29c)$$

$$\begin{aligned} \Xi_{kl}^{ij} = & \frac{Eh}{2bL^2} \int_0^b \int_0^L \left\{ \int_0^L \left[\sum_{i=1}^m \sum_{k=1}^n C_{ik}(t) \tilde{W}_i(y) \frac{\partial W_k(x')}{\partial x'} \right]^2 dx' \right\} \\ & \times \tilde{W}_i(y) \tilde{W}_j(y) \frac{\partial^2 W_k(x)}{\partial x^2} W_l(x) dx dy \end{aligned} \quad (29d)$$

where δ_{ij} and δ_{kl} are Kronecker deltas. For a supersonic flow, we can simply replace quantities R , Θ , and Υ in Eq. (29) by R_{super} , Θ_{super} , and Υ_{super} given in Eqs. (10) and (16), respectively. Similarly, for a subsonic flow, we can replace these quantities by R_{sub} , Θ_{sub} , and Υ_{sub} given by Eqs. (12) and (19), respectively.

Physically, the matrices $[\Phi]$, $[\Psi]$, $[\chi]$, and $[\Xi]$ in Eq. (28) reflect the effects of mass, damping, and stiffness per unit area of the plate, respectively. In particular, the quadruple integrals involved in the elements of the sub-matrices Φ_{kl}^{ij} , Ψ_{kl}^{ij} , and χ_{kl}^{ij} represent the effects of added mass, added damping, and added stiffness per unit area due to the radiated acoustic pressure field, respectively. The elements Ξ_{kl}^{ij} are nonlinear and contain quadratic

It is noted that the eigenfunctions $W_k(x)$ are orthogonal to each other

$$\int_0^L W_k(x)W_l(x) dx = \begin{cases} L, & k = l; \\ 0, & k \neq l. \end{cases} \quad (24)$$

However, the products of $W_k(x)$ and its derivatives are not necessarily orthogonal.

Similar relationships also hold true for $\tilde{W}_i(y)$

$$\int_0^L \tilde{W}_i(y)\tilde{W}_j(y) dy = \begin{cases} b, & i = j; \\ 0, & i \neq j. \end{cases} \quad (25)$$

Next, we expand $w(x, y, t)$ in terms of the orthogonal base functions $\{W(x)\}$ and $\{\tilde{W}(y)\}$ which satisfy the boundary conditions, Eq. (2),

$$w(x, y, t) = \{W(x)\}^T [C(t)] \{\tilde{W}(y)\} \quad (26)$$

where $\{W(x)\}$ and $\{\tilde{W}(y)\}$ represent column vectors containing the normal modes $W_k(x)$ and $\tilde{W}_i(y)$, respectively. A superscript T in Eq. (26) indicates a transposition, and $[C(t)]$ represents the matrix consisting of the unknown coefficients associated with $\{W(x)\}$ and $\{\tilde{W}(y)\}$. Physically, the j th column vector $\{C^j\}$ represents the amplitude of coupling between all the normal modes in the x -axis direction and the j th mode in the y -axis direction.

To facilitate the derivation of solution formulation, we also expand $p_T(x, y, 0, t)$, which represents the forcing field due to a turbulent boundary layer, by $\{W(x)\}$ and $\{\tilde{W}(y)\}$

$$p_T(x, y, 0, t) = \{W(x)\}^T [f(t)] \{\tilde{W}(y)\} \quad (27)$$

where $[f(t)]$ represents the amplitude of the excitation forcing field.

Substitute Eqs. (26) and (27) into (20), multiply the resulting equation by $\{W(x)\}$, integrate over x from 0 to L , and then multiply the equation by $\{\tilde{W}(y)\}^T$ and integrate over y from 0 to b . This procedure then leads to the following matrix equation

$$[\Phi][\ddot{C}] + [\Psi][\dot{C}] + [\chi][C] = [f], \quad (28)$$

The quantities a_{sub} and b_{sub} in Eq. (17) are given by

$$\Theta_{sub} = \frac{1}{c(1-M^2)} \left[\frac{(x-x')}{R_{sub}} - M \right], \quad (19a)$$

$$\Upsilon_{sub} = \frac{(x-x')}{R_{sub}^2}. \quad (19b)$$

Substituting Eq. (4) into (1), we can rewrite the plate equation as

$$\begin{aligned} D \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 w(x, y, t) - N_x(y, t) \frac{\partial^2 w(x, y, t)}{\partial x^2} + \gamma \frac{\partial w(x, y, t)}{\partial t} + \rho_p h \frac{\partial^2 w(x, y, t)}{\partial t^2} \\ = -p_T(x, y, 0, t) - p(x, y, 0, t), \end{aligned} \quad (20)$$

where p_T and p are fully defined.

II. GALERKIN'S METHOD

Equation (20) represents a nonlinear, integral-differential equation. Analytical solutions to such an equation cannot be found in general. Hence in this section, we seek approximate solutions via Galerkin's method.

Let $W_k(x)$ be the k th eigenfunction for a clamped plate in the longitudinal (x -axis) direction and $\tilde{W}_i(y)$ be the i th eigenfunction in the lateral (y -axis) direction given by

$$W_k(x) = \sigma_k [\sin(\lambda_k x/L) - \sinh(\lambda_k x/L)] + [\cos(\lambda_k x/L) - \cosh(\lambda_k x/L)], \quad (21a)$$

$$\tilde{W}_i(y) = \sigma_i [\sin(\lambda_i y/b) - \sinh(\lambda_i y/b)] + [\cos(\lambda_i y/b) - \cosh(\lambda_i y/b)], \quad (21b)$$

where σ_k is the k th modal ratio defined as

$$\sigma_k = \frac{\sin \lambda_k + \sinh \lambda_k}{\cos \lambda_k - \cosh \lambda_k}, \quad (22)$$

where λ_k is the corresponding k th eigenvalue determined by

$$\cos \lambda_k \cosh \lambda_k = 1. \quad (23)$$

For example, $\lambda_1 = 4.73$, $\lambda_2 = 7.8532$, and $\lambda_3 = 10.9956$. For a large k , $\lambda_k \rightarrow (2k+1)\pi/2$.

Interchanging the order of integrations with respect to the frequency and spatial domains, and carrying out the integration with respect to ω first, we obtain

$$\begin{aligned}
p(x, y, 0, t) = & \frac{\rho}{(2\pi)^2} \int_0^b \int_0^L \left[\frac{(1 + U\Theta_{super})}{R_{super}\sqrt{M^2 - 1}} \frac{\partial^2 w(x', y', t')}{\partial t'^2} \right] \Big|_{t'=\tau} dx' dy' \\
& + \frac{\rho U}{(2\pi)^2} \int_0^b \int_0^L \left[\frac{\Upsilon_{super}}{R_{super}\sqrt{M^2 - 1}} \frac{\partial w(x', y', t')}{\partial t'} - \frac{(1 + U\Theta_{super})}{R_{super}\sqrt{M^2 - 1}} \frac{\partial^2 w(x', y', t')}{\partial x' \partial t'} \right] \Big|_{t'=\tau} dx' dy' \\
& - \frac{\rho_0 U^2}{(2\pi)^2} \int_0^b \int_0^L \left[\frac{\Upsilon_{super}}{R_{super}\sqrt{M^2 - 1}} \frac{\partial w(x', y', t')}{\partial x'} \right] \Big|_{t'=\tau} dx' dy', \tag{14}
\end{aligned}$$

for supersonic flow. The integrands in Eq. (14) are to be evaluated at the retarded time $\tau = t - \Delta t_{super}$, with Δt defined as

$$\Delta t_{super} = k[M(x - x') + R_{super}]/(M^2 - 1), \tag{15}$$

The quantities Θ_{super} and Υ_{super} in Eq. (14) are given by

$$\Theta_{super} = \frac{1}{c(M^2 - 1)} \left[\frac{(x - x')}{R_{super}} - M \right], \tag{16a}$$

$$\Upsilon_{super} = \frac{(x - x')}{R_{super}^2}. \tag{16b}$$

Similarly, for subsonic flow we have

$$\begin{aligned}
p(x, y, 0, t) = & \frac{\rho}{(2\pi)^2} \int_0^b \int_0^L \left[\frac{(1 + U\Theta_{sub})}{R_{sub}\sqrt{1 - M^2}} \frac{\partial^2 w(x', y', t')}{\partial t'^2} \right] \Big|_{t'=\tau} dx' dy' \\
& + \frac{\rho U}{(2\pi)^2} \int_0^b \int_0^L \left[\frac{\Upsilon_{sub}}{R_{sub}\sqrt{1 - M^2}} \frac{\partial w(x', y', t')}{\partial t'} - \frac{(1 + U\Theta_{sub})}{R_{sub}\sqrt{1 - M^2}} \frac{\partial^2 w(x', y', t')}{\partial x' \partial t'} \right] \Big|_{t'=\tau} dx' dy' \\
& - \frac{\rho_0 U^2}{(2\pi)^2} \int_0^b \int_0^L \left[\frac{\Upsilon_{sub}}{R_{sub}\sqrt{1 - M^2}} \frac{\partial w(x', y', t')}{\partial x'} \right] \Big|_{t'=\tau} dx' dy', \tag{17}
\end{aligned}$$

where $\tau = t - \Delta t_{sub}$ with Δt_{sub} defined as

$$\Delta t_{sub} = k[M(x - x') + R_{sub}]/(1 - M^2). \tag{18}$$

where the symbol $\hat{\cdot}$ over a function, say, ϕ implies a Fourier transformation of that function defined by

$$\phi(x, y, t) = \int_{-\infty}^{\infty} \hat{\phi}(x, y, \omega) e^{i\omega t} d\omega, \quad (8a)$$

$$\hat{\phi}(x, y, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x, y, t) e^{-i\omega t} dt. \quad (8b)$$

The quantity G in Eq. (7) is a Green's function. For a supersonic flow, the acoustic pressure is nonzero only when the field point \vec{x} is inside the Mach cone. Hence we obtain⁶

$$G(x, y, z|x', y', 0) = -\frac{i2\pi \cos\{ik[M(x-x') + R_{super}]/(M^2-1)\}}{R_{super}\sqrt{M^2-1}} \quad (9)$$

where R_{super} is the distance between the source and receiver inside the Mach cone

$$R_{super} = \sqrt{(x-x')^2 - (M^2-1)[(y-y')^2 + z^2]} \quad (10)$$

Accordingly, G for a subsonic flow can be written as

$$G(x, y, z|x', y', 0) = -\frac{i2\pi e^{ik[M(x-x') + R_{sub}]/(1-M^2)}}{R_{sub}\sqrt{1-M^2}} \quad (11)$$

where

$$R_{sub} = \sqrt{(x-x')^2 + (1-M^2)[(y-y')^2 + z^2]} \quad (12)$$

where M is the Mach number of the mean flow.

The frequency domain solution for the radiated acoustic pressure is then given by

$$\hat{p}(x, y, z, \omega) = -\rho \left(-i\omega + U \frac{\partial}{\partial x} \right) \hat{\phi}(x, y, z, \omega). \quad (13)$$

The corresponding solution for the acoustic pressure on the plate surface in the time domain $p(x, y, 0, t)$ can be obtained by taking an inverse Fourier transformation of \hat{p} .

where p_T represents a forcing field which exerts the same amount of pressure fluctuations on the structure as the turbulent boundary layer, and p depicts the effect of acoustic radiation. Here we assume that the forcing field p_T is random in nature, but stationary in time and homogeneous in space. Therefore p_T can be described by a space-time cross-correlation function and determined experimentally. The function p satisfies the convected wave equation outside the boundary layer. Solution for p can be facilitated by introducing a potential function which satisfies⁹

$$\left[\nabla^2 - \frac{1}{c^2} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \right] \phi(x, y, z, t) = 0, \quad (5)$$

subject to the condition of continuity of normal components of the fluid particle velocities at, strictly speaking, the interface between the turbulent boundary layer and mean flow. However, it is impossible to determine this interface exactly because the flow beneath it is turbulent. To simplify the problem, we set this interface at the plate's nominal position so that the boundary condition becomes

$$\frac{\partial \phi(x, y, z, t)}{\partial z} \bigg|_{z=0} = \begin{cases} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) w(x, y, t) & \vec{x} \in A; \\ 0 & \text{otherwise;} \end{cases} \quad (6)$$

where A stands for the plate surface area, U is the mean flow speed, and c is the speed of sound of the fluid medium.

Solution for ϕ can be obtained by taking a temporal Fourier transformation of Eqs. (5) and (6) into the frequency domain, followed by a spatial Fourier transformation into the frequency-wavenumber domain. Once the frequency-wavenumber domain solution that represents an outgoing wave is obtained, the frequency domain solution of $\hat{\phi}$ can be obtained by taking an inverse spatial Fourier transformation

$$\hat{\phi}(x, y, z, \omega) = -\frac{i}{(2\pi)^2} \int_0^b \int_0^L G(x, y, z | x', y', 0) \left(-i\omega + U \frac{\partial}{\partial x'} \right) \hat{w}(x', y', \omega) dx' dy', \quad (7)$$

subject to the boundary conditions

$$w(0, y, t) = w(L, y, t) = w(x, 0, t) = w(x, b, t) = 0, \quad (2a)$$

$$\left. \frac{\partial w}{\partial x} \right|_{x=0} = \left. \frac{\partial w}{\partial x} \right|_{x=L} = \left. \frac{\partial w}{\partial y} \right|_{y=0} = \left. \frac{\partial w}{\partial y} \right|_{y=b} = 0. \quad (2b)$$

In Eq. (1), the quantity $D = Eh^3/12(1 - \mu^2)$ is the plate flexural rigidity, E the Young's modulus, μ the Poisson's ratio of the material, h the thickness, ρ_p the mass density, γ the structural viscous damping coefficient, w the plate flexural displacement in the z -axis direction, and N_x the stress resultant in the x -axis direction due to stretching of plate bending motion¹⁵

$$N_x(y, t) = \frac{Eh}{2L} \int_0^L \left[\frac{\partial w(x, y, t)}{\partial x} \right]^2 dx. \quad (3)$$

The effects of the stress resultant in the y -axis direction N_y and that in the tangential direction N_{xy} are assumed small compared with N_x , and are therefore neglected in Eq. (1).

The term on the right side of Eq. (1) depicts the excitation forcing field due to turbulent flow. Inside the turbulent boundary layer, the quantity \tilde{p} can be expressed as an integral representation,¹⁶ with a volume integral representing the contribution from distributed quadrupoles, plus a surface integral representing the effect of sound reflection and diffraction on the surface.¹⁷ However, the stress tensor¹⁸ in the volume integration contains quantities that are unknown until the entire flow field is solved, which is not possible in most engineering applications. Hence, a different approach must be taken.

Note that the effect of turbulent flow is to excite a structure and generate sound. Therefore we can assume that the pressure fluctuations acting on the plate surface \tilde{p} can be expressed as a sum of two parts

$$\tilde{p}(x, y, 0, t) = p_T(x, y, 0, t) + p(x, y, 0, t), \quad (4)$$

phenomenon to the physical quantities possessed by an elastic structure. For steady-state responses, we demonstrate that the cross-power spectral density function of the plate flexural displacement can be expressed as the cross-power spectral density function of turbulent flow (see Section IV). Based on the plate flexural vibration response, we can then derive a formulation for calculating the cross-power spectral density function of the radiated acoustic pressure (see Section V). Some concluding remarks are made in Section VI.

I. PLATE EQUATION

Assume that a plate of length L and width b is clamped on its edges to a rigid and infinitely extended baffle. On one side of the plate, the flow is turbulent, with boundary layer thickness δ . Outside this boundary layer, the fluid moves at mean velocity \vec{U} (see Fig. 1). The other side of the plate is assumed to be vacuum, so that the effect of fluid loading acts on one side of the plate only.

To carry out the analysis, we make the following assumptions: (1) the amplitude of the plate flexural vibration is small, so that the boundary layer will not be significantly altered; (2) the turbulent boundary layer is stationary in time and homogeneous in space, so that the pressure fluctuations can be expressed as a cross-correlation function which decays with spatial and temporal separations and convects downstream with the flow; and (3) the pressure fluctuations exerted on the plate surface due to a turbulent boundary layer and due to radiated acoustic pressure are additive.

Under these conditions, we can write the equation governing the plate flexural vibration as

$$D \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 w(x, y, t) - N_x(y, t) \frac{\partial^2 w(x, y, t)}{\partial x^2} + \gamma \frac{\partial w(x, y, t)}{\partial t} + \rho_p h \frac{\partial^2 w(x, y, t)}{\partial t^2} = -\tilde{p}(x, y, 0, t), \quad (1)$$

INTRODUCTION

Responses of elastic structures subject to turbulent and mean flow excitations have long been of great practical engineering importance, for example, in the design of aircraft and in the reduction of aerodynamic noise transmission into an aircraft cabin. Experimental and theoretical investigations on turbulent flow and its interaction with an elastic structure have been carried out by many people, for example, Ludwig¹, Corcos^{2,3}, Maestrello^{4,5}, Maestrello and Linden⁶, Farabee and Geib⁷, Martin and Leehey⁸, Yen et al.⁹, Ffowcs Williams¹⁰, Gedney and Leehey¹¹, Farabee and Casarella¹², etc. Most of the early work, however, has focused on the behavior of linear systems. Recently, with the development of modern computational facilities, nonlinearities in an elastic structure and in acoustic wave propagation can be accounted for, and the dynamic response of an elastic structure subject to a turbulent boundary layer excitation can be calculated directly^{13, 14}. Nevertheless, the physics involved in interaction between turbulent flows and vibrating structures is still not well understood, and usually cannot be revealed by direct numerical simulations.

In this paper, we derive an analytical formulation for calculating the dynamic and acoustic responses of a finite plate clamped flush to an infinite baffle under the excitation of turbulent flow, with the structural nonlinearities due to stretching of in-plane force taken into account (see Section I). Because of the presence of nonlinearities and coupling of structural modes to the radiated acoustic pressure, closed form solutions cannot be found. In order to obtain an approximate solution, the plate flexural displacement is expanded into the orthogonal base functions in the plate's longitudinal and transverse directions. The unknown coefficients associated with these base functions are determined by Galerkin's method. These are done in Section II. In Section III, we present an analysis of structural instabilities induced by fluid loading and flow. The objective of this stability analysis is to gain physical insight into this complex problem by relating the instability